Instituto Tecnológico de Buenos Aires

# Quasi-analytical Perturbation Analysis of the Generalized Nonlinear Schrödinger Equation

P. I. Fierens
pfierens@itba.edu.ar

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# Collaboration

J. Bonetti
S. M. Hernandez
D. F. Grosz

Instituto Balseiro

E. Temprana

University of California San Diego

# Generalized Nonlinear Schrödinger Eq.

**Pulse propagation in single-mode nonlinear fibers**

$$\frac{\partial A}{\partial z} - i\hat{\beta}A = i\hat{\gamma}A(z,T)\left|A(z,T)\right|^2.$$

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**Dispersion** $$\hat{\beta} = \sum_{k\geq 2} \frac{i^k \beta_k}{k!} \frac{\partial^k}{\partial T^k}$$

**Nonlinearity** $$\hat{\gamma} = \sum_{k\geq 0} \frac{i^k \gamma_k}{k!} \frac{\partial^k}{\partial T^k}$$

# Analytical solutions to the GNLSE

## Inverse-scattering

- Solitonic solutions (Zakharov 1972)
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## Akhmediev breathers

- Family of periodic solutions (Akhmediev and Korneev 1986)
- Integrability in more complex cases, but still a limited number of solutions

# Analytical solutions to the GNLSE

### Why do we care?

- Exact solutions of simplified versions provide important insight
- They cannot give a precise description in general $\Rightarrow$ the GNLSE is usually studied by means of simulations

# Our work

## A particular case

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## Didn't I talk about this 2 years ago?

- Yes! @ Denver: first order linear perturbation (modulation instability - MI)
- Problem 1: undepleted pump $\Rightarrow$ short distances
- Problem 2: disregards cascading four-wave mixing effect

# Pump + Noise

## Stationary solution + Perturbation

$$\frac{\partial \vec{\tilde{a}}(\zeta, \Omega)}{\partial \zeta} = \mathbf{H}(\Omega)\vec{\tilde{a}}(\zeta, \Omega) + \vec{\tilde{N}}(\vec{\tilde{a}}(\zeta, \Omega))$$

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**Normalized distance:** $\zeta = \gamma_0 P_0 \, z$

**Perturbation:** $A(\zeta,T) = \sqrt{P_0}\,[1 + a(\zeta,T)]\,e^{i\zeta}$

**Fourier transform:** $\vec{\tilde{a}}(\zeta,\Omega) = \begin{bmatrix} \tilde{a}(\zeta,\Omega) \\ \tilde{a}^*(\zeta,-\Omega) \end{bmatrix}$

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**Linear term:** $\mathbf{H}(\Omega) = i \begin{bmatrix} B(\Omega) & \tilde{\gamma}(\Omega) \\ -B(-\Omega) & -\tilde{\gamma}(-\Omega) \end{bmatrix}$

**Nonlinear term:** $\vec{\tilde{N}}(\vec{\tilde{a}}(\zeta,\Omega)) = \begin{bmatrix} \tilde{\gamma}(\Omega)\tilde{N}\left(\tilde{a}(\zeta,\Omega)\right) \\ \tilde{\gamma}(-\Omega)\tilde{N}^*\left(\tilde{a}(\zeta,\Omega)\right) \end{bmatrix}$

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$$\frac{\partial \vec{\tilde{a}}(\zeta,\Omega)}{\partial \zeta} = \mathbf{H}(\Omega)\vec{\tilde{a}}(\zeta,\Omega) + \vec{\tilde{N}}(\vec{\tilde{a}}(\zeta,\Omega))$$

- $B(\Omega) = \tilde{\beta}(\Omega) + 2\tilde{\gamma}(\Omega) - 1$
- $\tilde{\beta}(\Omega) = \frac{1}{\gamma_0 P_0}\sum_{m=2}^{M}\frac{(-1)^m}{m!}\beta_m\Omega^m$, $\tilde{\gamma}(\Omega) = \frac{1}{\gamma_0}\sum_{n=0}^{N}\frac{(-1)^n}{n!}\gamma_n\Omega^n$

$$\tilde{N}(\tilde{a}) = \left[\tilde{a}(\zeta,\Omega) * \bar{\tilde{a}}(\zeta,-\Omega)\right] + \tilde{a}(\zeta,\Omega) * \left[\tilde{a}(\zeta,\Omega) + \bar{\tilde{a}}(\zeta,-\Omega)\right] + \\ \tilde{a}(\zeta,\Omega) * \left[\tilde{a}(\zeta,\Omega) * \bar{\tilde{a}}(\zeta,-\Omega)\right]$$

# Modulation instability

## Linear perturbation analysis

$$\frac{\partial \vec{\tilde{a}}_1(\zeta, \Omega)}{\partial \zeta} = \mathbf{H}(\Omega)\vec{\tilde{a}}_1(\zeta, \Omega)$$

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## Motivation

This result motivates the following perturbative *ansatz*

# Higher-order perturbation

## Perturbative *ansatz*

$$\tilde{a}(\zeta,\Omega) \approx \sqrt{s} e^{i\phi_0(\zeta,\Omega)} + \sum_{n=1}^{\infty} \left( e^{G_n(\Omega)\zeta} - 1 \right) A_n(\Omega) \sqrt{s^n} e^{i\phi_n(\zeta,\Omega)}.$$

# Higher-order perturbation

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## Simplifying assumptions

- $\left\langle e^{i\phi_n(x,\mu)} \right\rangle = 0$
- $\left\langle e^{i(\phi_n(x,\mu) - \phi_m(y,\nu))} \right\rangle = 0$ if either $n \neq m$, $x \neq y$ or $\mu \neq \nu \longrightarrow$ similar to the 'random phase' hypothesis in optical wave turbulence (Picozzi et al. 2014)

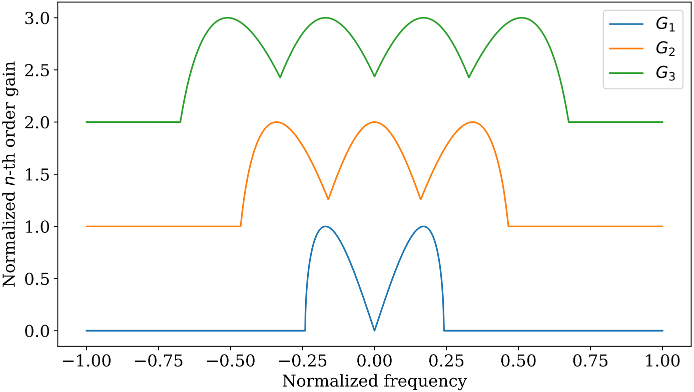
# Higher-order gain

## Gain

$$G_n(\Omega) \approx \max_{\mu} \left[G_1(\mu) + G_{n-1}(\Omega - \mu)\right]$$

- Arises from the convolutions in the nonlinear operator
- Largest gain dominates $\longrightarrow$ simplification of the convolution integrals
- Incorporates the gain due to the perturbations amplified by $G_n$ acting as $n$-th order 'pumps': cascading FWM effect

# Higher-order gain



## Perturbation spectrum

$$|A_1(\Omega)|^2 = \frac{\left(\frac{B(\Omega)+B(-\Omega)}{2}\right)^2 + G_1^2(\Omega) + \tilde{\gamma}^2(\Omega)}{2G_1^2(\Omega)}$$

$$|A_n(\Omega)| \approx \Delta_\Omega^{n-1} J(G_n(\Omega), \Omega)$$

$$J(g,\Omega) = \frac{\sqrt{\left|\overline{B}(-\Omega) - ig\right|^2 |\tilde{\gamma}(\Omega)|^2 + \left|\overline{C}(-\Omega)\right|^4}}{\left|[B(\Omega)+ig]\left[\overline{B}(-\Omega)-ig\right] - \tilde{\gamma}(\Omega)\tilde{\gamma}(-\Omega)\right|}$$

# Evaluation of the approximation

## Experiment

- A 770 m-long, dispersion-stabilized Highly-Nonlinear Fiber (Kuo et al. 2012)
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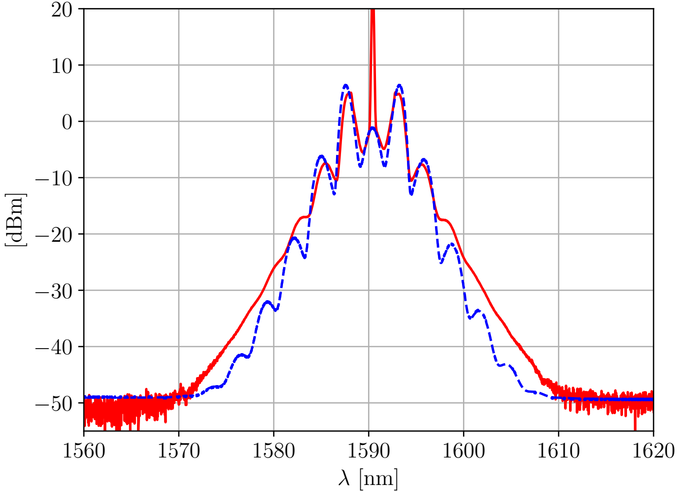
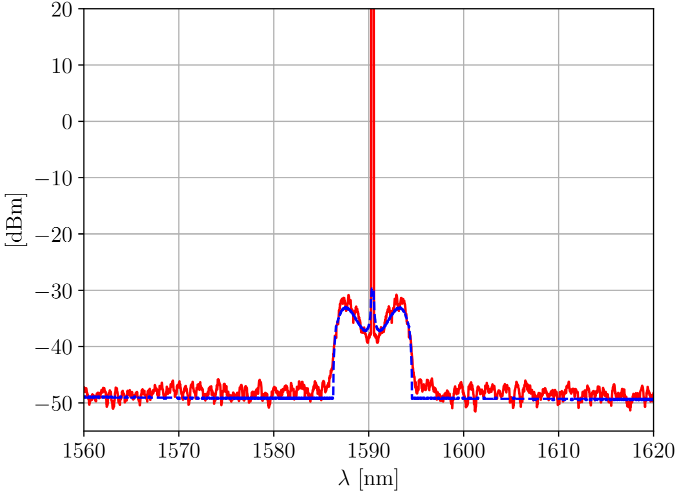
## Simulations

- Same parameters as the experiment
- $\gamma_0 = 8.7\ \mathrm{W}^{-1}\mathrm{Km}^{-1}$, $\beta_2 = -3.9198\ \mathrm{ps}^2/\mathrm{km}$, $\beta_3 = 0.1267$ $\mathrm{ps}^3/\mathrm{km}$, $\beta_4 = 1.7594 \times 10^{-4}\ \mathrm{ps}^4/\mathrm{km}$

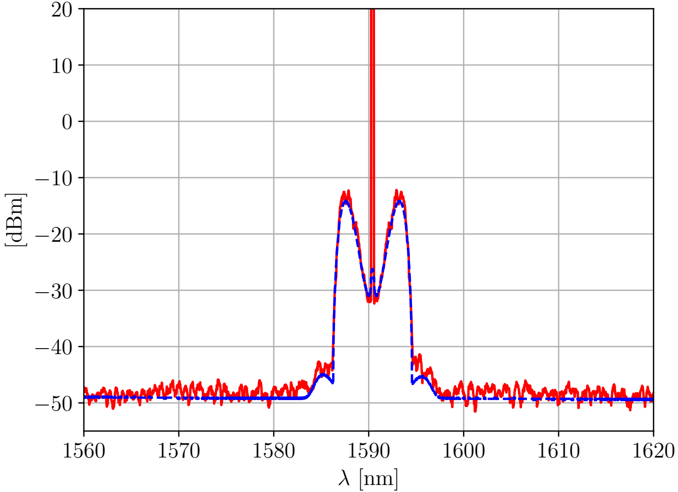


20
10
0
−10
−20
−30
−40
−50
[dBm]
1560
1570
1580
1590
1600
1610
1620
λ [nm]

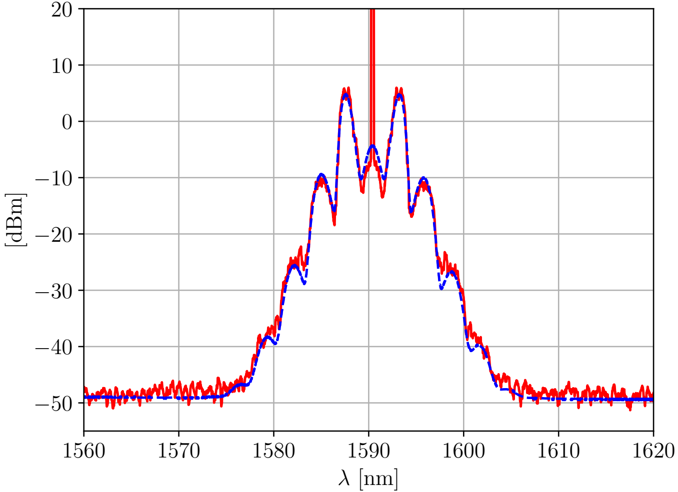
# Numerical results - 250 m



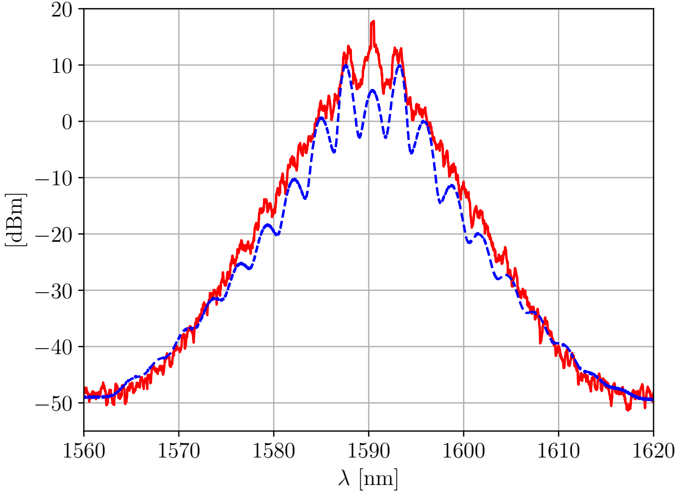
# Numerical results - 500 m



# Numerical results - 750 m



# Numerical results - 1000 m



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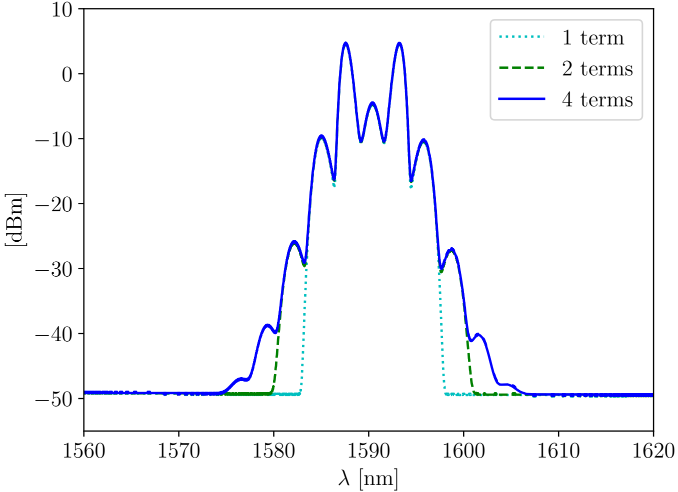
**Approximation order:** $n = 8$

**Higher order:** What is the influence of higher-order terms?

# Approximation orders



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**Simplification:** Is there a simple way to arrive to our results?


Thank You

# Bibliography I

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