Spectral dynamics of noise-seeded modulation instability

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Introduction

Modulation instability

Propagation of a CW in an optical fiber is unstable $\rightarrow$ breaks up into pulses

Vast literature

- and many more...
Rekindled interest in MI

**Supercontinuum generation**

Narrowband input to an optical fiber $\xrightarrow{\text{nonlinearity}}$ wideband signal


**Rogue waves**

High-amplitude and rare waves that ‘appear from nowhere’

Anything left unsaid?

40 years of research! Should I end my talk now?

- Most of the analyses of MI do not include all details relevant to optical fibers. One exception: Béjot et al. (2011)
- Not a lot of work on (quasi-)analytical approaches to the interaction of noise and nonlinearity in MI

In this talk

- A complete analysis of the spectral evolution of a perturbation to a CW
- First analytical results on input noise + MI
- Some recent results on a noisy input
- Ongoing work...
Applications

What are we interested in?

- Supercontinuum generation in the mid IR
  - Molecular fingerprint region
  - Applications in metrology, tomography, isotope separation, ...
  - Lack of sources in some bands
- Intense pulses - rogue waves
- Parametric amplification
Spectral dynamics of noise-seeded modulation instability

Propagation in optical fibers

Generalized nonlinear Schrödinger equation

\[ \frac{\partial A}{\partial z} - i\hat{\beta}A = i\hat{\gamma}A(z, T) \int_{-\infty}^{+\infty} R(T') |A(z, T - T')|^2 dT', \tag{1} \]

- Dispersion:
  \[ \hat{\beta} = \sum_{m \geq 2} \frac{i^m}{m!} \beta_m \frac{\partial^m}{\partial T^m} \]

- Nonlinearity:
  \[ \hat{\gamma} = \sum_{n \geq 0} \frac{i^n}{n!} \gamma_n \frac{\partial^n}{\partial T^n} \]

- Raman scattering: \[ R(T') = (1 - f_R)\delta(T) + f_R h(T) \]
Perturbation to the stationary solution

\[ A(z, T) = \left( \sqrt{P_0} + a \right) e^{i\gamma_0 P_0 z} = A_s + ae^{i\gamma_0 P_0 z} \]

- **Input power:** \( P_0 \)
- **Perturbation:** \( a(z, T) \)

Linear terms in the frequency domain

\[ \frac{\partial \tilde{a}(z, \Omega)}{\partial z} + \tilde{N}(\Omega) \tilde{a}(z, \Omega) = \tilde{M}(\Omega) \tilde{a}^*(z, -\Omega), \]

- **Frequency:** \( \Omega = \omega - \omega_0 \)
- **\( \tilde{N}(\Omega) \):**
  \[ \tilde{N}(\Omega) = -i \left[ \tilde{\beta}(\Omega) + P_0 \tilde{\gamma}(\Omega) \left( 1 + \tilde{R}(\Omega) \right) - P_0 \gamma_0 \right] \]
- **\( \tilde{M}(\Omega) \):**
  \[ \tilde{M}(\Omega) = iP_0 \tilde{\gamma}(\Omega) \tilde{R}(\Omega) \]
Perturbation to the stationary solution

Ansatz: \( a(z, \Omega) = D \exp(iK(\Omega)z) \)

\[
K(\Omega) = -\tilde{B}(\Omega) \pm K_D(\Omega) = -\tilde{B}(\Omega) \pm \sqrt{\tilde{B}^2(\Omega) - \tilde{C}(\Omega)}
\]

- \( \tilde{B}(\Omega) \) and \( \tilde{C}(\Omega) \) are complex functions of the parameters
- Agrees with Béjot \textit{et al.} (2011)
Modulation-instability gain

Only self-steepening: $\gamma_1 = \gamma_0 \tau_{sh}$, $\gamma_n = 0$ for $n \geq 2$

$$K(\Omega) = \tilde{\beta}_o + P_0 \gamma_0 \tau_{sh} \Omega (1 + \tilde{R}) \pm \sqrt{\left(\tilde{\beta}_e + 2\gamma_0 P_0 \tilde{R}\right) \tilde{\beta}_e + P_0^2 \gamma_0^2 \tau_{sh}^2 \Omega^2 \tilde{R}^2}.$$ 

- Well-known facts about MI gain $= 2\text{Im}\{K(\Omega)\}$:
  - It does not depend on odd terms of the dispersion relation
  - Self-steepening enables a gain even in a zero-dispersion fiber
  - In the large power limit, it is independent of the dispersion and it is dominated by Raman:

$$|g(\Omega)| \approx 2P_0 \gamma_0 \tau_{sh} |\Omega| \cdot |\text{Im } \tilde{R}(\Omega)|.$$
Pump power: 100 W (left) and 5 kW (right)

With Raman ($G_3$ and $G_4$) and with self-steepening ($G_2$ and $G_4$)
Spectral evolution

Spectrum

\[ \tilde{a}(z, \Omega) = e^{-i\tilde{B}(\Omega)z} \frac{\tilde{M}(\Omega) \sin(K_D(\Omega)z) \tilde{a}^*(0, -\Omega) + e^{-i\tilde{B}(\Omega)z} \left[ K_D(\Omega) \cos(K_D(\Omega)z) - \left( \tilde{N}(\Omega) - i\tilde{B}(\Omega) \right) \sin(K_D(\Omega)z) \right] \tilde{a}(0, \Omega)}{K_D(\Omega)} \]

- Interaction between \( \tilde{a}(0, \Omega) \) and \( \tilde{a}(0, -\Omega) \) due to nonlinearity
- \( a(0, T) \in \mathbb{R} \iff \tilde{a}(z, \Omega) = \tilde{H}(\Omega, z)\Lambda(\Omega) \)
Noise-only input

White Gaussian noise

\[ \tilde{a}(0, \Omega) \sim \mathcal{CN}(0, \sigma^2) \rightarrow \tilde{a}(z, \Omega) \sim \mathcal{CN}(0, \sigma_{\tilde{a}}^2) \]

\[ \rightarrow |\tilde{a}(z, \Omega)| \sim \text{Rayleigh}(\sigma_{\tilde{a}}) \rightarrow |\tilde{a}(z, \Omega)|^2 / \sigma_{\tilde{a}}^2 \sim \chi_2^2 \]

\[ \sigma_{\tilde{a}}^2 = \sigma^2 \left| \frac{e^{-i\tilde{B}(\Omega)z}}{K_D(\Omega)} \right|^2 \left\{ |\tilde{M}(\Omega) \sin(K_D(\Omega)z)|^2 + \right. \]

\[ + \left. |K_D(\Omega) \cos(K_D(\Omega)z) - (\tilde{N}(\Omega) - i\tilde{B}(\Omega)) \sin(K_D(\Omega)z)|^2 \right\}. \]
Noise-only input

@ 10 mm (top) and @ 40 mm (bottom)
Noise-only input

Histograms of $|\tilde{a}(z, \Omega)|$ for $z = 10$ mm

$f = 26.758$ THz (left) and $f = -26.758$ THz (right)
Two metrics from supercontinuum generation

Coherence - Dudley, Coen and Genty (2006)

\[ g_{12}(z, \Omega) = \frac{\langle \tilde{a}^*_k(z, \Omega)\tilde{a}_l(z, \Omega) \rangle_{k \neq l}}{\sqrt{\langle |\tilde{a}_k(z, \Omega)|^2 \rangle \langle |\tilde{a}_l(z, \Omega)|^2 \rangle}} \]

- In our setting: \( g_{12}(z, \Omega) = 0 \)

Signal-to-noise ratio - Sørensen et al. (2012)

\[ \text{SNR}(\Omega) = \frac{\langle |\tilde{a}(z, \Omega)|^2 \rangle}{\sqrt{\text{Var} \ (|\tilde{a}(z, \Omega)|^2)}} \]

- In our setting: \( \text{SNR}(\Omega) = 1 \)
Textbook case

Tractable example as a sanity check

\( \beta_2 < 0 \) (anomalous dispersion), \( \beta_k = 0 \) for \( k > 2 \), \( \gamma_n = 0 \) for \( n > 0 \), \( \tilde{R}(\Omega) = 1 \)

\[
\sigma_{\tilde{a}}^2 = \begin{cases} 
\sigma^2 \left\{ 1 + \left[ \frac{2 (\frac{\Omega_c}{\Omega})^4}{\left( \frac{\Omega_c}{\Omega} \right)^2 - 1} \right] \sinh^2 \left( \frac{z |\beta_2| \Omega^2}{2} \sqrt{\left( \frac{\Omega_c}{\Omega} \right)^2 - 1} \right) \right\} & \Omega < \Omega_c \\
\sigma^2 \left\{ 1 + \left[ \frac{2 (\frac{\Omega_c}{\Omega})^4}{1 - \left( \frac{\Omega_c}{\Omega} \right)^2} \right] \sin^2 \left( \frac{z |\beta_2| \Omega^2}{2} \sqrt{1 - \left( \frac{\Omega_c}{\Omega} \right)^2} \right) \right\} & \Omega > \Omega_c 
\end{cases}
\]

\( \Omega_c = 4 \gamma_0 P_0 / |\beta_2| \)
Textbook case

Approximation for large \( z \)

\[
\sigma_{\tilde{a}}^2 \approx \sigma^2 \left\{ 1 + \alpha z \left[ e^{-\frac{(\Omega - \Omega z)^2}{W_z}} + e^{-\frac{(\Omega + \Omega z)^2}{W_z}} \right] \right\} \rightarrow
\]

\[
r_a(z, \tau) \approx \sigma^2 \left\{ \delta(\tau) + \frac{8 \sinh^2 \left( \frac{z}{L_{NL}} \right)}{\sqrt{\pi} |\beta_2| z} e^{-\frac{\tau^2}{4|\beta_2|^2 z}} \cos \left( \frac{\Omega_c}{\sqrt{2}} \tau \right) \right\}.
\]

\[
L_{NL} = (\gamma_0 P_0)^{-1}
\]

- **periodicity** → breakup of the CW pump into pulses with a period

\[
\approx \sqrt{2}/\Omega_c
\]
Noisy input

Additive white Gaussian noise

\[ a(0, \Omega) = \tilde{s}(\Omega) + \eta(\Omega), \quad \eta(\Omega) \sim \mathcal{CN}(0, \sigma_a^2) \]

- Relevant for controlling the generation of rogue waves - Solli, Ropers & Jalali (2008); Dudley, Genty & Eggleton (2008); Sørensen et al. (2012)
- We developed analytical expressions for the coherence and SNR of the resulting spectrum (accepted at Phys. Rev. A)
Seeded coherence

Seed frequencies: $\Omega_1 = 31$ GHz and $\Omega_2 = 46$ GHz
Seeded coherence

A simple case: One-sided seed, $\tilde{s}(\Omega) = 0$ for $\Omega < 0$

- No self-steepening and no Raman: $\gamma_n = 0$ for $n \geq 1$, $\tilde{R}(\Omega) = 1$
- Net MI gain: $g(\Omega) = 2\text{Im}\{K_D(\Omega)\}$

$z \ll L_{NL}$

$$g_{12}(z, \Omega) \approx \begin{cases} 
1 - \left(1 + \left(\frac{z}{L_{NL}}\right)^2\right) \frac{\sigma^2}{|\tilde{s}(|\Omega|)|^2} & \Omega > 0, \\
1 - \left(2 + \left(\frac{L_{NL}}{z}\right)^2\right) \frac{\sigma^2}{|\tilde{s}(|\Omega|)|^2} & \Omega < 0.
\end{cases}$$

$g(\Omega)z \gg 1$

$$g_{12}(z, \Omega) \approx 1 - 2 \frac{\sigma^2}{|\tilde{s}(|\Omega|)|^2}$$
Conclusions

So far

- Analytical expressions for the spectral evolution of a perturbation to a continuous pump propagating in an optical fiber, including all relevant effects
- First steps in the analysis of the interaction of noise with the nonlinearity
  - Analytical results for some metrics of supercontinuum generation, such as coherence, for noisy inputs

Ongoing work

- Beyond the undepleted pump approximation
- Influence of seeding on the generation of coherent rogue waves
Questions?
References

- Akhmediev & Korneev, Theoretical and Mathematical Physics 69(2), 1089 (1986)
- Dudley, Genty & Eggleton, Optics Express 16, 3644 (2008)
- Shabat & Zakharov, Soviet Physics JETP 34, 62 (1972)
- Solli et al., Nature 450(7172), 1054 (2007)
Perturbation to the stationary solution

Ansatz: \( a(z, \Omega) = D \exp(iK(\Omega)z) \)

\[
K(\Omega) = -\tilde{B}(\Omega) \pm K_D(\Omega) = -\tilde{B}(\Omega) \pm \sqrt{\tilde{B}^2(\Omega) - \tilde{C}(\Omega)}
\]

\[
\tilde{B}(\Omega) = -\left[ \tilde{\beta}_o(\Omega) + P_0\tilde{\gamma}_o(\Omega) (1 + \tilde{R}(\Omega)) \right]
\]

\[
\tilde{C}(\Omega) = \tilde{\beta}_o^2(\Omega) - \tilde{\beta}_e^2(\Omega) + P_0^2 (\tilde{\gamma}_o^2(\Omega) - \tilde{\gamma}_e^2(\Omega)) (1 + 2\tilde{R}(\Omega)) - P_0^2 \gamma_0^2 + 2P_0\gamma_0\tilde{\beta}_e(\Omega) + 2P_0^2\gamma_0\tilde{\gamma}_e(\Omega) (1 + \tilde{R}(\Omega)) + 2P_0 \left( \tilde{\beta}_o\tilde{\gamma}_o - \tilde{\beta}_e\tilde{\gamma}_e \right) (1 + \tilde{R}(\Omega))
\]

\[
\tilde{\beta}_e(\Omega) = \sum_{n \geq 1} \frac{\beta_{2n}}{(2n)!} \Omega^{2n}
\]

\[
\tilde{\beta}_o(\Omega) = \sum_{n \geq 1} \frac{\beta_{2n+1}}{(2n+1)!} \Omega^{2n+1}
\]

\[
\tilde{\gamma}_e(\Omega) = \sum_{n \geq 0} \frac{\gamma_{2n}}{(2n)!} \Omega^{2n}
\]

\[
\tilde{\gamma}_o(\Omega) = \sum_{n \geq 0} \frac{\gamma_{2n+1}}{(2n+1)!} \Omega^{2n+1}
\]